Quiz 2 - Math 210

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Note: In all the exam, $\mathbb{R}$ is endowed with the Standard metric and every subset A of $\mathbb{R}$ with the restriction of that metric to $A \times A$.

Exercise 1. Let f be the function defined on $\mathbb{R}\setminus\{3\}$ by $f(x) = \frac{1}{x-3}$. Show, using the $\epsilon-\delta$ definition, that f is continuous on $\mathbb{R} \setminus \{3\}$.

Exercise 2. Let (X, d) be a metric space. Show that any closed ball in (X, d) is closed in (X, d).

Exercise 3. Let $K := \{\frac{1}{n}; n \in \mathbb{N}^*\} \cup \{0\}$. Show that K is a compact subset of $\mathbb{R}$ using two different methods; one of them being the initial definition (i.e. using open covers).

Exercise 4. (Boundary of a set)
Let $A \subseteq \mathbb{R}$. We say that a point $x \in \mathbb{R}$ is a *boundary point* of A if

$$\forall \epsilon > 0, (x-\epsilon, x+\epsilon) \cap A \neq \emptyset \quad \textbf{and} \quad (x-\epsilon, x+\epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset.$$

We denote by ∂A the subset of $\mathbb{R}$ consisting of boundary points of A.

Recall that $\overline{A}$ denotes the closure of A in $\mathbb{R}$, A' denotes the set of limit points of A in $\mathbb{R}$ and $\overset{\circ}{A}$ the set of its interior points.

1. For each subset A of the following list, determine the sets A', $\overline{A}$ and ∂A.
   (a) $A = (a, b)$ with $a, b \in \mathbb{R}$ such that $a < b$. **No justification needed**.
   (b) $A = (0, 1) \cup \{2, 3\}$. **Brief justification <u>only</u> for** ∂A.
   (c) $A = \mathbb{Q} \cap [0, 1]$. **Brief justification <u>only</u> for** ∂A.
2. Let $A \subseteq \mathbb{R}$. Show that $\partial A = \overline{A} \setminus \overset{\circ}{A} = \{x \in \mathbb{R}; x \in \overline{A} \text{ and } x \notin \overset{\circ}{A}\}$.
3. Let $A \subseteq \mathbb{R}$. Show that A is closed in $\mathbb{R}$, if and only if, it contains all of its boundary points (i.e. $\partial A \subseteq A$).
4. Let $A \subseteq \mathbb{R}$. Let f be the characteristic function of A i,.e.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \end{aligned}$$

   Show that f is continuous at a point $x_0 \in \mathbb{R}$, if and only if, $x_0 \notin \partial A$.

Exercise 5. (Power Series)
Consider a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.

1. Show that, if the series $\sum_{n=0}^{\infty} a_n b^n$ converges for some real number b, then the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for every real number x such that $|x| < |b|$.

2. Let
$$S := \left\{ |x|; \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\} \qquad \text{and} \qquad R := \sup S.$$
Note that $0 \in S$, so S is not empty and $R \geq 0$ (eventually $+\infty$).

   Let $x \in \mathbb{R}$. Show that the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.

Consequences and Terminology:

- We conclude that the set $I := \{x \in \mathbb{R}; \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}$ is an interval centered at 0 whose endpoints are $\pm R$.
- We conclude also easily that R is the unique number in $[0, +\infty]$ that satisfies the conclusion of Question 2.
- The formal expression $\sum_{n=0}^{\infty} a_n x^n$ is called a *Power Series*, the number R is called the radius convergence of the power series and I is called its interval of convergence. Moreover, this power series defines a function from I to $\mathbb{R}$.

3. (Hadamard formula) Show that
$$R = \frac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}}.$$
Convention: $\frac{1}{0} = +\infty$ *and* $\frac{1}{+\infty} = 0$.

4. Deduce that the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also R.

5. Show that the function $f(x) := \sum_{n=0}^{+\infty} a_n x^n$ is continuous on the interior of I, i.e. on $(-R, R)$.
Hint: Recall that for every $x, y \in \mathbb{R}$, and for every $n \geq 1$, $|y^n - x^n| \leq nc^{n-1}|y - x|$, where $c := \max\{|x|, |y|\}$.

Exercise 1.

1. Consider the function $f(x) = \frac{1}{x-3}$ defined on $\mathbb{R} \setminus \{3\}$ (the latter being endowed with the restricted metric). Let $x_0 \neq 3$ and $\epsilon > 0$. Take

$$\delta := \min\left\{\frac{|x_0-3|}{2}, \epsilon\frac{|x_0-3|^2}{2}\right\}.$$

Consider an arbitrary real number x such that $x \neq 3$ and $|x - x_0| < \delta$. First, note that by the (reversed) triangular inequality,

$$|x-3| = |(x-x_0)-(x_0-3)| = |(x_0-3)-(x-x_0)| \geq ||x_0-3| - |x-x_0|| \geq |x-0-3|-|x-x_0|.$$

Since $|x - x_0| < \delta$ and $\delta < \frac{|x_0-3|}{2}$, we deduce that

$$|x-3| > |x_0-3| - \frac{|x_0-3|}{2} = \frac{|x_0-3|}{2}.$$

Hence,

$$|f(x) - f(x_0)| = \left|\frac{1}{x-3} - \frac{1}{x_0-3}\right| = \frac{|x-x_0|}{|x-3|\,|x_0-3|} < \frac{2\delta}{|x_0-3|^2}.$$

Since $\delta < \epsilon\frac{|x_0-3|^2}{2}$, we deduce that

$$|f(x) - f(x_0)| < \epsilon.$$

Hence f is continuous at x_0. This being true for an arbitrary $x_0 \neq 3$, we deduce that f is continuous on $\mathbb{R} \setminus \{3\}$.

Exercise 2. Let (X, d) be a metric space, $x_0 \in X$ and $\epsilon > 0$. The closed ball in (X, d) of center x_0 and radius ϵ is the following subset of X:

$$A := B'(x_0, \epsilon) = \{x \in X; d(x, x_0) \leq \epsilon\}.$$

We will prove that $X \setminus A$ is open in (X, d). Let $x \in X \setminus A$, i.e. $d(x, x_0) > \epsilon$. Put

$$\delta := d(x, x_0) - \epsilon.$$

Note that $\delta > 0$ as $x \notin A$. We claim that $B(x, \delta) \cap A = \emptyset$ or equivalently $B(x, \delta) \subseteq X \setminus A$. Indeed, consider an arbitrary element $y \in B(x, \delta)$. Then by the triangular inequality (or the reversed one),

$$d(y, x_0) > d(x, x_0) - d(x, y).$$

As $y \in B(x, \delta)$, $d(x, y) < \delta$ so that

$$d(y, x_0) > d(x, x_0) - \delta = \epsilon.$$

Consequently, $y \notin A$. Since y was arbitrary in $B(x, \delta)$, we conclude that

$$B(x, \delta) \subseteq X \setminus A.$$

Hence x is an interior point of $(X \setminus A)$. Since x was arbitrary in $X \setminus A$, we deduce that $X \setminus A$ is an open subset of (X, d), or equivalently, A is closed in (X, d).

Exercise 3. Let $K = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$. Let us prove that K is a compact subset of $\mathbb{R}$. We propose fours methods, only the forth being the one using the initial definition of compactness using open covers. The first three are basically equivalent (at least in their proof).

1. Method 1: By Heine Borel's theorem, its sufficient and necessary to prove that K is bounded and closed in $\mathbb{R}$. Indeed, K is clearly bounded for the Standard metric as $|x| \leq 1$ for every $x \in K$. Moreover, if $(x_n)_{n\in\mathbb{N}}$ is sequence of elements in K that converges to some $l \in \mathbb{R}$, then either

   - $l = 0$ and this happens if $(x_n)_{n\in\mathbb{N}}$ is a subsequence of the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ or if $x_n = 0$ eventually
   - or $l = \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}^*$, and this happens if $x_n = \frac{1}{n_0}$ eventually

   . In any case $l \in K$; so that any sequence of elements of K can converge only in K. Hence K is closed in $\mathbb{R}$. Compactness of K follows from Heine-Borel's theorem.

   Remark: another method for showing that K is closed in $\mathbb{R}$ is to write

   $$\mathbb{R} \setminus K = (-\infty, 0) \cup \bigcup_{n\in\mathbb{N}^*} (\frac{1}{n}, \frac{1}{n}) \cup (1, +\infty)$$

   and then conclude that $\mathbb{R} \setminus K$ is open in $\mathbb{R}$ (as being union of open intervals) and hence that K is closed in $\mathbb{R}$.

2. Method 2 (Alternative of Method 1): K is a compact subset of $\mathbb{R}$ if and only if K is sequentially compact. Let then $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in K. As in Method 1, one shows that it admits a subsequence that converges either 0 or to $\frac{1}{n_0}$ for some $n_0 \in \mathbb{N}^*$, and hence it admits a subsequence that converges in K. Hence K is compact.

3. Method 3 (Another alternative of Method 1, proposed indirectly by some students)
   By Sheet 5, Ex. 8, we know (I am not assuming that you know, but this is just a method) that if A is a closed subset of $\mathbb{R}$ which is included in some compact subset B of $\mathbb{R}$, then A is also compact. Here K is a closed subset of $\mathbb{R}$ (proof done in Method 1) and $K \subset [0, 1]$ which is compact (as being closed and bounded). Hence K is compact.

4. Method 4: Using initial definition with open covers
   Consider an arbitrary cover $\mathcal{Y} := \{U_i; i \in I\}$ of K by open subsets of $\mathbb{R}$. Since $0 \in K$, then there exists some $i_0 \in I$ such that $0 \in U_{i_0}$. Since U_{i_0} is open in $\mathbb{R}$, there exists some $\epsilon > 0$ such that $(-\epsilon, +\epsilon) \subseteq U_{i_0}$. Let $N = \lceil \frac{1}{\epsilon} \rceil$. For every $n \geq N$, $-\epsilon < 0 < \frac{1}{n} < \epsilon$; so that

   $$\forall n \geq N, \frac{1}{n} \in U_{i_0}.$$

   Now for each $k \in \{1, \cdots, N-1\}$, $\frac{1}{k} \in K$ so that there exists some index $i_k \in I$ such that $\frac{1}{k} \in U_{i_k}$. Consequently,

   $$K \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_{N-1}} \cup U_{i_0}.$$

   Hence we extracted from the cover $\mathcal{Y}$ of K the subcover $\mathcal{Y}' := \{, U_{i_0}, U_{i_1}, U_{i_2}, \cdots, U_{i_N}\}$ which contains finitely many elements. Hence, K is compact.

Exercise 4. Note that for every $x \in \mathbb{R}$,

$$x \in \partial A \iff x \text{ is adherent to } A \text{ and to } \mathbb{R} \setminus A.$$

In other terms,

$$\partial A = \overline{A} \cap \overline{\mathbb{R} \setminus A}.$$

1. (a) Let $A = (a, b)$. Then $\overline{A} = [a, b]$, $\overset{\circ}{A} = A = (a, b)$, $A' = [a, , b]$, $\partial A = \{a, b\}$.

   (b) Let $A = [0, 1) \cup \{2, 3\}$. Then $\overline{A} = [0, 1] \cup \{2, 3\}$, $\overset{\circ}{A} = (0, 1)$, $A' = [0, 1]$, $\partial A = \{0, 1, 2, 3\}$.

   Justification for the boundary:

   i. First, we check that $\{0, 1, 2, 3\} \subseteq \partial A$. As these points belong to A, the first condition required for being a boundary point is clearly fulfilled (i..e that fact that every open neighborhood about the point intersects A). It is left to prove that for each of them, every open neighborhood around it intersect $\mathbb{R} \setminus A$. Let $\epsilon > 0$. We have $(-\epsilon, \epsilon) \cap (\mathbb{R} \setminus A) \supset [-\frac{\epsilon}{2}, 0) \neq \emptyset$, $(1 - \epsilon, 1 + \epsilon) \supset (\min\{1 + \frac{\epsilon}{2}, 1.9\}, 2) \neq \emptyset$, $(2 - \epsilon, 2 + \epsilon) \cap (\mathbb{R} \setminus A) \supset \big(\max\{2 - \frac{\epsilon}{2}, 1.1\}, 2\big) \neq \emptyset$ and $(3 - \epsilon, 3 + \epsilon) \cap (\mathbb{R} \setminus A) \supset (3, 3 + \frac{\epsilon}{2}) \neq \emptyset$.

   ii. Now we check that $\partial A \subseteq \{0, 1, 2, 3\}$, or equivalentally that if $\mathbb{R} \setminus \{0, 1, 2, 3\} \cap \partial A = \emptyset$. Indeed, the set $\{0, 1, 2, 3\}$ is closed in $\mathbb{R}$ (as being a finite set), so that its complement in $\mathbb{R}$ is open. Hence every element x of $\mathbb{R} \setminus \{0, 1, 2, 3\}$ is an interior point of that set, i.e. there exists some ϵ_0 such that $(x - \epsilon_0, x + \epsilon) \cap (\mathbb{R} \setminus A) = \emptyset$. Hence $x \notin \partial A$.

   (c) Let $A = \mathbb{Q} \cap [0, 1]$. Then $\overline{A} = [0, 1]$. $A' = [0, 1]$, $\overset{\circ}{A} = \emptyset$ and $\partial A = [0, 1]$.

   Justification for the boundary:

   i. First, we check that $[0, 1] \subseteq \partial A$. The deep reason behind this fact is that $\mathbb{Q}$ is a subset of $\mathbb{R}$ which is dense but with empty interior, or equivalently it is dense and its complement in $\mathbb{R}$ is also dense.
   Let $x \in [0, 1]$ and $\epsilon > 0$. If $x \in \mathbb{Q}$, then definitely $(x - \epsilon, x + \epsilon) \cap \mathbb{Q} \supset \{x\} \neq \emptyset$. Moreover, by density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, $(x - \epsilon, x + \epsilon) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$. Hence $x \in \partial A$ in this case. Now if $x \in \mathbb{Q}$, then using this time the density of $\mathbb{Q}$ in $\mathbb{R}$, we get that $x \in \partial A$. In any case $x \in \partial A$.

   ii. Now we prove that $(\mathbb{R} \setminus [0, 1]) \cap \partial A = \emptyset$. Indeed, $\mathbb{R} \setminus [0, 1]$ is open in $\mathbb{R}$ (why?), so every point x of $(\mathbb{R} \setminus A)$ it is an interior point and hence cannot be a boundary point (as there is some neighborhood of x that does not intersect A). This proves our claim.

2. We already observed that for every $x \in \mathbb{R}$,

$$x \in \partial A \iff x \in \overline{A} \text{ and } x \in \overline{\mathbb{R} \setminus A}.$$

   However,

$$x \in \overline{\mathbb{R} \setminus A} \iff \forall \epsilon > 0, (x - \epsilon, x + \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset \iff \forall \epsilon > 0, (x - \epsilon, x + \epsilon) \not\subset A \iff x \notin \overset{\circ}{A}.$$

   Hence

$$x \in \partial A \iff x \in \overline{A} \text{ and } x \notin \overset{\circ}{A}.$$

   In other terms, $\partial A = \overline{A} \setminus \overset{\circ}{A}$.

3. Assume that A to be closed in $\mathbb{R}$ and take an arbitrary boundary point x of A in $\mathbb{R}$. This is in particular an adherent point to A. Since A is closed in $\mathbb{R}$, we deduce that $x \in A$.

   Conversely, assume that A contains all of its boundary points and take an arbitrary adherent point x of A. We will prove that $x \in A$. If $x \in \overset{\circ}{A}$, then we're done as $\overset{\circ}{A} \subseteq A$. If not, then $x \in \overline{A} \setminus \overset{\circ}{A}$. By Question 2, this implies that $x \in \partial A$. Since we assumed $\partial A \subseteq A$, we deduce that $x \in A$. In both cases, $x \in A$. Hence A contains all of its boundary points (i.e. $\overline{A} \subseteq A$), or equivalently, A is closed in $\mathbb{R}$. This proved the desired result.

4. Let $x_0 \in \mathbb{R}$. First, we explain what becomes the $\epsilon - \delta$ definition of continuity for our particular function f. The simplification will be due to the fact that $\text{Im}(f) = \{0, 1\}$. Actually, the following characterization of continuity is true for any function (defined on any metric space) taking values 0 or 1.

$$
\begin{aligned}
f \text{ is continuous at } x_0 \quad &\iff \quad \forall \epsilon > 0, \exists \delta > 0; \forall x \in B(x_0, \delta), |f(x) - f(x_0)| < \epsilon && (1)\\
&\iff \quad \forall 0 < \epsilon < 1, \exists \delta > 0; \forall x \in B(x_0, \delta), |f(x) - f(x_0)| < \epsilon && (2)\\
&\iff \quad \exists \delta > 0; \forall x \in B(x_0, \delta), |f(x) - f(x_0)| = 0 && (3)\\
&\iff \quad \exists \delta > 0; \forall x \in B(x_0, \delta), f(x) = f(x_0) && (4)\\
&\iff \quad f \text{ is constant on some open neighborhood of } x_0
\end{aligned}
$$

The equivalence of statements (1) and (2) is a general trivial (but useful) fact (convince yourself why). Also (3) $\implies$ (2) is trivial. The key point is implication (2) $\implies$ (3). Suppose that (2) holds. Note that Im(f) $= \{0, 1\}$ so that for any $a, b \in \mathbb{R}$, $|f(a) - f(b)| \leq 1$. Applying now (2) for $\epsilon = 0.5$ (for instance), we deduce immediately (3).

We are now ready to end the proof.

- Suppose that $x_0 \in \partial A$. Then every neighborhood around x_0 contains an element of A and an element of $\mathbb{R} \setminus A$; so every open neighborhood around x_0 contains an element x such that $f(x) \neq f(x_0)$. By (4), f is discontinuous at x_0.
- Suppose now that f is discontinuous at x_0. By (4), every neighborhood around x_0 should contain an element x such that $f(x) \neq f(x_0)$. Hence every neighborhood around x_0 will contain an element of A and $\mathbb{R} \setminus A$ (if $x_0 \in A$ choose x_0 for the element in A and x for the element in $\mathbb{R} \setminus A$ and the opposite if $x_0 \notin A$). Hence, we proved that if f was discontinuous at x_0 then $x_0 \in \partial A$.

This ends the proof.

Exercise 4.
Consider a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.

1. Let $b \in \mathbb{R}$ such that $\sum_{n=0}^{+\infty} a_n b^n$ converges. If $b = 0$, then the required result is trivial. Suppose then that $b \neq 0$ and consider a real number x such that $|x| < |b|$. Since $b \neq 0$, we can write for every integer n,

$$|a_n x^n| = |a_n b^n| \times \left(\left|\frac{x}{b}\right|\right)^n.$$

   SInce the series $\sum_{n=0}^{\infty} a_n b^n$ converges, then its nth term $a_n b^n$ tends to 0. In particular,

$$\exists n_0 \in \mathbb{N}; \forall n \geq n_0, |a_n b^n| < 1.$$

   Hence

$$\forall n \geq n_0, |a_n x^n| < \left(\left|\frac{x}{b}\right|\right)^n.$$

   Observe now that since $|x| < |b|$, the series $\sum_{n=0}^{+\infty} (|x|/|b|)^n$ is a geometric series with ratio < 1 in absolute value, hence it converges. By the Direct Comparison Test (for positive sequence), we deduce that the series $\sum_{n=0}^{+\infty} |a_n x^n|$ converges, i.e. the series $\sum_{n=0}^{+\infty} a_n x^n$ converges absolutely (and in particular it converges).

2. Consider a real number x such that $|x| > R$. Then by definition of S, the series $\sum_{n=0}^{+\infty} a_n x^n$ diverges. Suppose now that $|x| < R$. Since $R = \sup S$, then there exists some $b \in S$ such that $|x| < b \leq R$. By definition of S, we have that the series $\sum_{n=0}^{\infty} a_n b^n$ converges. But since $|x| < b$, Question 1 tells us that the series $\sum_{n=0}^{+\infty} a_n x^n$ converges absolutely and is in particular convergent. This is what we wanted to prove.

3. Let $\rho := \frac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}}$. We want to prove that $\rho = R$. We will do it by proving that ρ satisfies the conditions of Question 2, which is enough to conclude (by uniqueness of R for these properties). Fix $x \in \mathbb{R}$ and let $z_n := a_n x^n$ for $n \in \mathbb{N}$. Suppose first that $|x| < \rho$. Then

$$\limsup_{n\to+\infty} \sqrt[n]{|z_n|} = |x| \times \limsup_{n\to+\infty} \sqrt[n]{|a_n|} = \rho x. \tag{5}$$

   Since $|x| < \rho$, we deduce from (5) that $\limsup\limits_{n\to+\infty} \sqrt[n]{|z_n|} < 1$. By the Root Test, we deduce that the series $\sum_{n=0}^{+\infty} z_n$ converges, i.e. that the series $\sum_{n=0}^{\infty} a_n x^n$ converges. Suppose now that $|x| > \rho$. We have by (5) that $\limsup\limits_{n\to+\infty} \sqrt[n]{|z_n|} > 1$, which implies by the Root test that the series $\sum_{n=0}^{+\infty} z_n$ diverges, i,e, that the series $\sum_{n=0}^{+\infty} a_n x^n$ diverges. Hence ρ satisfiers the condition of Question 2 and then $\rho = R$.

4. Let $a'_n = n a_n$, $n \in \mathbb{N}^*$. We have

$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = \limsup_{n\to+\infty} (n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}). \tag{6}$$

   But $n^{\frac{1}{n}} \underset{n\to+\infty}{\longrightarrow} 1$ (Basic Limits). It is an exercise to show that this fact, together with (6), imply that

$$\limsup \sqrt[n]{|c'_n|} = \limsup \sqrt[n]{|a_n|} = R.$$

   By the previous question, we deduce that the radius of convergence of the Power Series $\sum_{n=0}^{+\infty} c'_n x^n$ is $\mathbb{R}$, i.e. the same as the radius of convergence of the Power Series $\sum_{n=0}^{+\infty} a_n x^n$.

5. First, we recall a property. If $\sum_n c_n$ is a series that converges absolutely, then $\sum_n c_n$ converges and

$$\left|\sum_{n=0}^{\infty} c_n\right| \leq \sum_{n=0}^{\infty} |c_n|. \tag{7}$$

   Let now $x_0 \in (-R, R)$. We will prove that f is continuous on x_0. Consider another $x \in (-R, R)$ and let $c_x := \max\{|x|, |x_0|\}$. First note that, since x and x_0 are in the interior of I, the series $\sum_n a_n x^n$ and $\sum_n a_n x_0^n$ converge absolutely. This implies that the series $\sum_{n=0}^{\infty} a_n (x^n - x_0^n)$ is also absolutely convergent. Indeed, the nth general term $\eta_n := a_n(x^n - x_0^n)$ of this series satisfies

$|\eta_n| = |a_n(x^n - x_0^n)| \leq |a_n x^n| + |a_n x_0^n|$, and the series of general term $\gamma_n := |a_n x^n| + |a_n x_0^n|$ are convergent (sum of two convergent series); we conclude by DCT. Hence we can use inequality (7) for the series $\sum_{n=0}^{\infty} a_n(x^n - x_0^n)$. We have then by (7):

$$
\begin{aligned}
|f(x) - f(x_0)| &= \left| \sum_{n=0}^{+\infty} a_n x^n - \sum_{n=0}^{+\infty} a_n x_0^n \right| \\
&= \left| \sum_{n=1}^{+\infty} a_n (x^n - x_0^n) \right| && (8) \\
&\leq \sum_{n=1}^{+\infty} |a_n||x^n - x_0^n| && (9) \\
&\leq \sum_{n=1}^{+\infty} |a_n| n c_x^{n-1} |x - x_0| && (10) \\
&= |x - x_0| \sum_{n=1}^{+\infty} n|a_n| c_x^{n-1} && (11)
\end{aligned}
$$

The series $\sum_{n=1}^{+\infty} n|a_n|c_x^{n-1}$ in the last inequality is a series of positive terms ,so it converges in $[0, +\infty]$. Hence the passage from line (9) to line (10) has a meaning but has to be understood a priori in $[0, +\infty]$. However, since $|c_x| = \max\{|x|, |y|\} < R$, this sum is finite by Question 1.

Let $\delta := \frac{R - |x_0|}{2} > 0$ and assume from now on that $|x - x_0| < \delta$. This implies that $|x| < \frac{R + |x_0|}{2}$. Since $|x_0| < \frac{R+|x_0|}{2}$, we deduce that $c_x < \frac{R+|x_0|}{2} < R$. Hence

$$
|f(x) - f(x_0)| \leq |x - x_0| \sum_{n=1}^{+\infty} n|a_n| \left(\frac{R + |x|}{2} \right)^n \leq |x - x_0| \underbrace{\sum_{n=1}^{+\infty} n|a_n| \left(\frac{R + |x_0|}{2} \right)^n}_{M}
$$

Since $\frac{R+|x_0|}{2} < R$, we deduce that Question 1 that the series converges, i.e. that $M < +\infty$. Taking the limits in N, we deduce that

$$
|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < M\delta.
$$

This clearly implies that f is continuous on x_0 (for $\epsilon > 0$, choose $\delta := \epsilon / M_x$). Since $x_0 \in (-R, R)$ was arbitrary, we deduce that f is continuous on $(-R, R)$.

Important remarks:

1. Note that it was very important M depends ONLY on x_0 (and not on x).
2. It was also very important that the point x_0 was in the interior of I, i.e. $|x_0| < R$.
3. Remark about the passage in Question 4. If $(a_n)_n$ and $(b_n)_{n \in \mathbb{N}}$ are two sequences of real numbers SUCH THAT THE SEQEUNCE $(a_n)_n$ CONVERGES, then

$$
\limsup_{n \to +\infty} a_n b_n = (\lim_{n \to +\infty} a_n) \times \limsup_{n \to +\infty} b_n.
$$

In the case of Question 4, one can apply this equality for e $a_n = n^{\frac{1}{n}}$ and $b_n = |a_n|^{\frac{1}{n}}$ (notation of the Exercise).

In general, it is not true that $\limsup_{n \to +\infty} a_n b_n = \limsup_{n \to +\infty} a_n \limsup_{n \to +\infty} b_n$. If both sequences are non negative, then we have the following inequality:

$$
\limsup_{n \to +\infty} a_n b_n \leq \limsup_{n \to +\infty} a_n \limsup_{n \to +\infty} b_n.
$$

An example where the inequality is strict (for positive sequences) is the following: $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 1, \cdots$ and $b_0 = 1, b_1 = 0, b_2 = 0, b_3 = 1, \cdots$, we have $a_n b_n = 0$ for every $n \in \mathbb{N}$

so that $\limsup_{n\to+\infty} a_n b_n = 0 < 1 = (\limsup_{n\to+\infty} a_n) \times (\limsup_{n\to+\infty} b_n)$.

Now if the terms are non negative, the inequality above may not be true (why?)